PMM U.S.S.R.,Vol.52,No.4,pp.541-545,1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

THE FORMATION OF A SELFSIMILAR SOLUTION FOR THE PROBLEM OF NON-LINEAR WAVES IN AN ELASTIC HALF-SPACE*

A.P. CHUGAINOVA

Motions in the form of planar non-linear quasitransverse waves in a weakly non-isotropic elastic medium are studied. The selfsimilar problem of the action of a sudden load on the boundary of an elastic half-space has been considered previously /1-3/. The solution of the above-mentioned selfsimilar problem in a certain domain of the specified parameters was found to be non-unique (2 solutions). On account of this, the question arises as to in which cases one or the other solutions is realized in the domain of non-uniqueness. The question of the selection of the unique solution has been discussed in a number of papers /4-8/. In particular a condition for the existence of structure /6-8/ was required for the choice of the solution. An investigation of the structure of the discontinuities in the selfsimilar problem of the action of a suddenly applied load on the boundary of an elastic half-space /9/ showed that all the discontinuities occurring in the solutions (when there are two solutions) possess a structure and did not provide any grounds for preferring one solution over the other. In the present paper the selfsimilar asymptotic forms for a number of non-selfsimilar problems are found numerically in order to select the unique solution and these asymptotic forms are obtained in a similar manner to that used in /4/ in the case of problems in hydrodynamics.

In order to simplfy the calculations, use is made of the approximate equations /10/ which describe weakly non-linear quasitransverse waves in an elastic medium with a small anisotropy which propagate in only one direction.

$$\frac{\partial u_\alpha}{\partial t}+\frac{\partial}{\partial x}\left(\frac{\partial R}{\partial u_\alpha}-\frac{1}{2\sqrt{\mu\rho_0}}\tau_{\alpha 3}\right)=0,\quad \alpha=1,2 \tag{1}$$

$$R(u_1,\ u_2)={}^1/_2(f-g)u_1^2+{}^1/_2(f+g)u_2^2-{}^1/_8(u_1^2+u_2^2)^2\varkappa_1$$

$$u_i=\partial w_i/\partial x,\ \tau_{\alpha 3}=\rho_0\nu\partial u_\alpha/\partial t\approx\nu\sqrt{\mu\rho_0}\partial u_\alpha/\partial x$$

Here w_i are the displacements of the particles which are considered as a function of the Lagrangian coordinates, $x_1, x_2, x_3=x, \tau_{\alpha 3}$ are the components of the viscous stress tensor, ρ_0 is the density in the unstressed state, μ is the Lamé elastic coefficient, ν is the kinematic coefficient of viscosity, $\varkappa_1$ is a constant with the dimensions of velocity which characterizes the non-linear effects, f and g are constants and, moreover, g is a (small) anisotropy parameter, and f is the characteristic velocity when there is no anisotropy and non-linearity. The function R plays the role of an elastic potential. System (1) contains two equations for the shear components of the deformations, and the longitudinal components of the deformations u_3 are expressed in terms of the shear components (as in /10/).

A numerical calculation was carried out for different values of the coefficients $f, g, \varkappa_1$ and ν. However, by passing to a moving coordinate system x^*, t^*, where $x^*=x+ft, t^*=t$, and, correspondingly, introducing a change in the scales for the quantities u_1, u_2, x^* and t^*, system (1) can be reduced to a form such that $f=0, g=\varkappa_1=\nu=1$. Hence, the solutions which are obtained and described below can be recalculated to give the solutions of the universal equations obtained in this manner and vice versa.

The simple waves and shock waves (SW) which correspond to the simplified equations without dissipative terms and the structures of the shock waves are in agreement to an acceptable degree of accuracy with those obtained in /1-3, 9/ where the exact equations describing the propagation of non-linear, quasitransverse waves were taken as the initial equations but approximate methods of solution were used.

It has been shown /3/ in the case of the selfsimilar problem of a sudden change of the load on the boundary of an elastic half-space that two solutions exist in a certain domain of values of u_1^* and u_2^* for small values of the expression $2g/[(U^2+V^2)\varkappa_1]$

Below, we present the results of the numerical solution of a number of initial-boundary value problems for Eqs.(1) with viscous terms in the case of which the above-mentioned selfsimilar solutions can exhibit asymptotic forms as $t\to\infty$. Eqs.(1) were written in the form of implicit non-linear difference equations to which Newton's method was initially applied

*Prikl.Matem.Mekhan.,52,4,692-697,1988

followed by matrix pivotal condensation /11/. The calculation was carried out in an x, t domain bounded by a segment of the x-axis and fixed right and left boundaries.

The waves contained in the solution have different velocities of propagation and, therefore, as the computational time interval t between the individual perturbations increases, segments will appear, the length of which grows as t increases, which correspond to constant values of u_1 and u_2, while the structure of the SW as $t \to \infty$ will tend to become stationary. Hence, in order to identify the waves which are obtained from the numerical calculation with the waves occurring in the composition of selfsimilar solutions of the first or second kinds, it is necessary to obtain a difference solution at large times t in order to successfully complete the above-mentioned processes. On the other hand, as t increases, the difference solution is propagated onto a large number of mesh points. On account of this, it is necessary that a sufficiently large interval of the x-axis be taken in order that the effect of the right-hand boundary should not distort the solution. The effect of the left boundary on the solution is removed by a corresponding choice of the coefficient f which was chosen in such a way that perturbations were propagated to the right.

The numerically constructed shock adiabatic curve $APQALC$ (the initial point A (1, 1)) is shown in Fig.1. The points J and E are Jouguet points while the points Q and P depict states such that the shock waves $A \to Q, A \to P$ propagate at the same velocity as the shock wave $A \to J$. The domain where the selfsimilar solution (a value of 0.1 was adopted for the anisotropy factor $2g/\varkappa_1$) is non-unique is hatched in. This domain is bounded by the interval PE of the shock adiabatic from the initial point A, the evolutionary interval of the shock adiabatic QP, constructed from the point Q as the initial point and by the intervals EZ_1 and QZ_2 of the integral curves of the simple undisturbed waves.

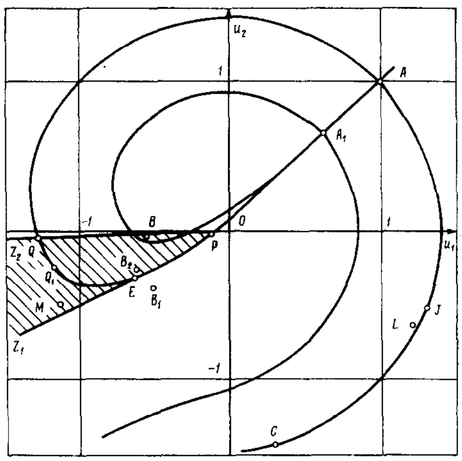

Fig.1

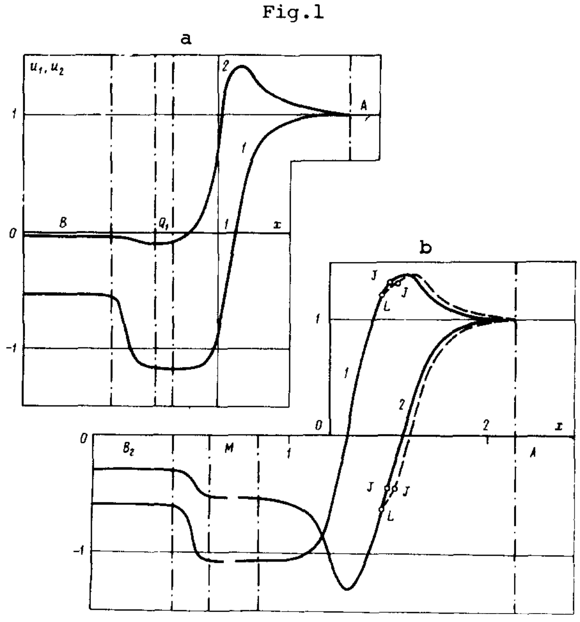

Fig.2

In one of the versions of the calculation, the initial conditions $(t=0)$ and the condition on the right boundary $(t>0, x=l)$ were taken in the form $u_1=1, u_2=1$ (point A). The values $u_1=-0{,}574$ and $u_2=-0.038$ were taken on the left boundary $(t>0,\ x=0)$ (point B from the domain where the solution is non-unique). The results of the numerical calculation for a certain instant of time are shown in Fig.2, a where it is possible to pick out the shock waves which are identified with the shock waves which have been considered theoretically: a sequence of two shock waves (from their structure) consisting of a fast and a slow shock wave (the transition from point A to point Q_1 and from point Q_1 to point B) **Figs.1 and 2a.**

The intervals of the x-axis (which are picked out by the dotted and dashed lines in Fig.2,a) correspond to the points A, B and Q_1 in Fig.1. The values of u_1 and u_2 take constant values in these intervals. It was checked that these constant values which represent states in front of and behind the shock waves $A\to Q_1$ and $Q_1\to B$ satisfy the equations of the corresponding shock adiabatics with an acceptable accuracy. It should be noted that, within the structure which corresponds to the shock transition from the point A to the point Q_1, the magnitude of u_2 has a local maximum (Fig.2,a). This is in accord with a qualitative investigation /9/ of the solutions representing the structure of the corresponding shock waves.

Similar versions of the calculation were also carried out for other values of u_1 and u_2 on the left boundary of the x, t domain which correspond to points from the domain of non-uniqueness, the coordinates of which were:

u_1^*	-0.3	-0.4	-0.5	-0.6	-0.7	-0.8	-1	-1	$-1{,}1$
u_2^*	-0.1	-0.2	-0.2	-0.3	-0.35	-0.3	-0.3	-0.34	-0.25

A solution of the first type was realized in all cases.

The calculations showed that a solution of the second type for values of u_1 and u_2 on the left boundary which correspond to points from the domain of non-uniqueness can be formed in the following manner. First, it is necessary to form completely a solution of the second type for u_1 and u_2 on the left boundary which correspond to a point from the region of uniqueness. In order to do this the coordinates u_1 and u_2 of a point, which does not belong to the domain of non-uniqueness (point B_1 in Fig.1) but which lies fairly close to the boundary of this region, were taken as the condition on the left boundary. The values of u_1 and u_2 on the left boundary were then changed to values which belong to the domain of non-uniqueness (point B_2 in Fig.1). This leads to a state of affairs where small perturbations will propagate from the left boundary to the right. After waves which have previously been formed interact with these perturbations, the solution at large times must emerge onto a certain selfsimilar asymptotic form. A calculation has shown that it does not change its type.

The results of the calculation are shown in Fig.2,b where it is possible to pick out a sequence of simple waves and shock waves which completely correspond to a solution of the second type: a fast Jouguet shock wave, that is, the jump from point A to point J and then a fast simple wave, that is, the interval of the integral curve JL, a fast Jouguet shock wave (the jump from L to M) and, then, the jump of the slow shock wave from M to B_2. The segment JL, which corresponds to a simple wave has been separated out in Fig.2,b by comparison of the solutions at two successive instants of time. The graph of the solution for the instant of time t_1 is depicted by the solid line and, for the instant of time $t_2>t_1$, by the broken line on that part of the curve where the graphs do not coincide at instants of time t_1 and t_2, and by a solid line where they do coincide. The interval JL broadens with the passage of time while the intervals which correspond to the structures of the jumps $A\to J$ and $L\to M$ do not change.

The intervals on the x-axis where the values of u_1 and u_2 are constant in Fig.2,b correspond to the points A, M and B_2 in Fig.1 while the interval of the curve, which corresponds to the point M in Fig.1, is depicted by the break. This means that some of the computed points are not shown and the values of u_1 and u_2 at these points are equal to u_1 and u_2 up to and after the break in the graph. The magnitude of u_1 has a local maximum within the structure of the jumps from the initial point A to the point J while the magnitude of u_2 has a local maximum in the structure of the jump from point M to point L. This is in agreement with the integral curves in /9/.

The results which have been obtained therefore show that, as $t\to\infty$, the solution of a non-selfsimilar problem may tend to the different solutions of the limiting selfsimilar problem (if its solutions is non-unique) depending on the details of the formulation of the initial and boundary conditions corresponding to bounded values of time.

It is impossible to subdivide explicitly all the initial and boundary conditions into two classes corresponding to the different asymptotic behaviour as $t\to\infty$. However, if we confine ourselves to a treatment of formulations of problems in which continuous functions with bounded derivatives occur, then, for the unique extension of the solutions with respect to time as $\nu\to 0$, it is necessary to know how to solve problems on the interaction of two shock waves or the interaction between a shock wave and a small perturbation (the collision of a large number of waves of low probability). In doing this, there is, of course, interest in cases when the corresponding selfsimilar solution is non-unique.

On account of this, the problem of the interaction of two shock waves, one of which catches up the other, was considered. For this purpose, a structure of the slow shock wave was formed which corresponds to the evolutionary jump from point A to point A_1 (Fig.1) and a structure for the fast shock wave which catches it up, which corresponds to the evolutionary jump from the point A_1 to point B (Fig.1). Point B is located on the shock adiabatic curve traced from the point A_1 as the initial point and belongs to the domain of non-uniqueness of the solution of the selfsimilar problem corresponding to point A as the initial point. A calculation showed that two other shock waves are formed as a result of the interaction of these two shock waves. A fast shock wave travels forward with a slow shock wave behind it and the solution therefore emerges onto the selfsimilar asymptotic form which corresponds to a solution of the first type of the selfsimilar problem of the change in a load on the boundary of an elastic half-space.

The problem of the interaction of a fast shock wave, corresponding to the evolutionary jump from point A to the point Q_1 (or any other point of the evolutionary interval EQ), with a small perturbation was also considered. The small perturbation was formed in front of the shock wave. The numerical calculations which were carried out showed that a single shock wave is formed as a result of the interaction of a shock wave and a small perturbation.

In accordance with the solution of the selfsimilar problem of a sudden change in the load on the boundary of an elastic half-space obtained in /3/, a numerical study was carried out into how the form of the solution changes when the points B_i, which represent the left boundary condition, lying close to the line EP which separates the domains of which a solution of the first and second kind is realized from above but a solution of the second type from below. For this purpose, we obtained the solution for a series of values of u_1 and u_2 on the left boundary, some of which correspond to points lying above the line EP and some which correspond to points lying below the line EP: $u_1^* = -0.5$, $u_2^* = -0.2, -0.25, -0.3, -0.33, -0.34, -0.345, -0.35$. A calculation showed that, in the case of points which depict the state on the left boundary and are located above a point with the coordinates (-0.5, -0.34), a solution of the first type is realized while, in the case of points lying below this point, a solution of the second type is realized.

Hence, the calculations show that the selfsimilar asymptotic forms of the first type (a sequence of a fast shock wave and a slow shock wave or a simple wave) are formed more stably (in a certain sense they possess a greater domain of attraction) than the selfsimilar asymptotic forms of the second type. In the case of problems with slowly varying initial and boundary conditions, the processes involving the interaction of the shock waves which arise in the solution will obviously take place just as if the second selfsimilar solution did not exist.

It is interesting to note that a solution of the first type, which arises most frequently in calculations, does not satisfy the stability condition with regard to the smoothing of the initial function /5, 12/. In essence, the latter condition is a stability condition with respect to finite perturbations and, as a numerical calculation shows, the fact that it is not satisfied does not lead in the case under consideration to any spontaneous reconstruction of the solution.

The author thanks A.G. Kulikovskii and E.I. Sveshnikova for formulating the problem and their remarks during a discussion of the work.

REFERENCES

1. SVESHNIKOVA E.I., Simple waves in a non-linear elastic medium, Prikl. Mat. i Mekh., 46, 4, 1982.
2. KULIKOVSKII A.G. and SVESHNIKOVA E.I., Investigation of the adiabatic shock curve of quasi-transverse shock waves in a previously stressed elastic medium, Prikl. Mat. i Mekh., 46, 5, 1982.
3. KULIKOVSKII A.G. and SVESHNIKOVA E.I., The selfsimilar problem of the action of a sudden load on the boundary of an elastic half-space, Prikl. Mat. i Mekh., 49, 2, 1985.
4. BARMIN A.A. and USPENSKII V.S., Development of pulsed regimes in one-dimensional non-stationary MHD-flows with cutoff of electric conductivity, Izv. Akad. Nauk SSSR, Mekh. Zhid. i Gaz., 4, 1986.
5. OLEINIK O.A., On a class of discontinuous solutions of first order quasinon-linear equations, Nauchn. Dokl. Vyssh. Shk. Fiz.-Mat. Nauki, 3, 1958.
6. GALIN G.YA., On shock waves in media with an arbitrary equation of state, Dokl. Akad. Nauk USSR, 119, 6, 1958.
7. GALIN G.YA., On the theory of shock waves, Dokl. Akad. Nauk SSSR, 127, 1, 1959.

8. KALASHNIKOV A.S., Construction of generalized solutions of first-order quasilinear equations without the condition of convexity as limits of the solutions of parabolic equations with a small parameter, Dokl. Akad. Nauk SSSR, 127, 1, 1959.
9. KULIKOVSKII A.G. and SVESHNIKOVA E.I., Investigation of the structure of quasitransverse elastic shock waves, Prikl. Mat. i Mekh., 51, 6, 1987.
10. KULIKOVSKII A.G., On the equations describing the propagation of non-linear quasitransverse waves in a weakly anisotropic elastic body, Prikl. Mat. i Mekh., 50, 4, 1988.
11. SAMARSKII A.A. and POPOV YU.P., Difference Methods for Solving Problems in Gas Dynamics, Nauka, Moscow, 1980.
12. ROZHDESTVENSKII B.L. and YANENKO N.N., Systems of Quasilinear Equations, Nauka, Moscow, 1978.

Translated by E.L.S.

PMM U.S.S.R., Vol.52, No.4, pp.545-547, 1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

TRANSIENT ANTIPLANE VIBRATIONS OF A RECTANGULAR ELASTIC SLAB*

N.V. BOYEV and M.A. SUMBATYAN

The non-stationary antiplane problem of an elastic rectangle under specified stresses on its lateral edge is considered. A solution of the problem in Laplace **transforms has been** obtained in the form of a series of homogeneous solutions. The use of certain special operator relationships enables one to write out the original of the solution in an explicit form. When this is done for any instant of time, each homogeneous solution is expressed in the form of a finite sum. A numerical analysis of the problem is presented and the characteristic features of the behaviour of the stressed state in time are established.

Let us consider the transient vibrations of an elastic slab of infinite length (the y-axis) and rectangular cross-section with sides of $2h$ and $2a$, $z \in [-h, h]$, $x \in [-a, a]$ under conditions of antiplane deformation caused by forces acting on opposite lateral edges $\xi = \pm 1$, $z \in [-h, h]$

$$\tau_{\xi y}(\xi, \zeta, T)|_{\xi=\pm 1} = f(\zeta, T);\ \xi = x/a,\ \zeta = z/h;\ \xi, \zeta \in [-1, 1] \tag{1}$$

Here $f(\zeta, T)$ is an arbitrary function of the variables ζ and the time T. For simplicity, let us assume that the edges $\zeta = \pm 1$ are free from stresses $(\tau_{\zeta y}|_{\zeta=\pm 1} = 0)$ and the initial conditions are:

$$v|_{T=0} = \partial v/\partial T|_{T=0} = 0,\ 1 < \xi,\ \zeta < 1.$$

$(v = v(\xi, \zeta, T)$ is the displacement along the y-axis).

Let us apply a Laplace transform with respect to time to the initial boundary value problem. We have

$$\varepsilon^2 \frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \zeta^2} = \varepsilon^2 p^2 V, \quad V = \int_0^\infty v(\xi, \zeta, t) e^{-st} dt$$

$$t = T/T_0,\ \varepsilon = h/a,\ p^2 = \rho a^2 s^2/(\mu T_0^2)$$

Here, ρ is the density of the material, μ is the shear modulus and T_0 is the characteristic time. Since, subject to condition (1), the function V is odd with respect to ξ, we shall seek it in the form $W(\zeta, p) \operatorname{sh} \gamma\xi$, where the function $W(\zeta, p)$ is determined from the following selfadjoint boundary value problem

$$\frac{d^2}{d\zeta^2} W(\zeta, p) = -\varepsilon^2 (\gamma^2 - p^2) W(\zeta, p), \quad \frac{d}{d\zeta} W(\pm 1, p) = 0 \tag{2}$$

In the case of a problem which is symmetric with respect to ζ, the eigenfunctions of problem (2) have the form

*Prikl.Matem.Mekhan., 52, 4, 697-699, 1988